AIAA JOURNAL
Vol. 33, No. 9, September 1995

Panel Method Formulation for Oscillating Airfoils in Supersonic Flow

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In the present work, the mathematical model for two-dimensional unsteady supersonic flow based on the reduced potential equation, also known as the hyperbolic Helmholtz equation, is presented and discussed. The purpose is to develop a rigorous formulation, in order to bring into light the correspondence between the supersonic and subsonic panel method theory. Source and doublet integrals are obtained, and Laplace transformation demonstrates that these are both solutions to the complete problem. It is shown that the doublet-only formulation reduces to a Volterra integrodifferential equation of the second kind, and a numerical method is proposed in order to solve it. To the authors' knowledge, this is the first reported solution to the unsteady supersonic thin airfoil problem through the use of doublet singularities. Comparisons with the source-only method are shown for the problem of a flat plate in harmonic heave. Results for the problems of a flat plate in harmonic pitch and under a sinusoidal gust are also presented. The doublet panel method was used to assess the importance of considering unsteady terms in the solution, as opposed to applying quasisteady formulations.

Nomenclature

a	= undisturbed speed of sound
C_l	= complex lift coefficient
C_{l1}, C_{l2}	= real and imaginary parts of C_l
C_m	= complex moment coefficient
C_{m1}, C_{m2}	= real and imaginary parts of C_m
C_p	= complex pressure coefficient
$h(x)$	= airfoil surface curve, upper or lower
h_0	= heaving motion amplitude
J_0, J_1	= zeroth- and first-order Bessel functions of the first kind
K	= hyperbolic Helmholtz equation frequency parameter, defined by $k_r M/(M^2 - 1)$
k_r	= reduced frequency, given by $\omega L/U$
L	= reference length, equal to half the airfoil chord
L_1, L_2	= real and imaginary parts of $C_l/(4k_r^2)$, for a flat plate in unitary amplitude heave
L_3', L_4'	= real and imaginary parts of $C_l/(4k_r^2)$, for unitary amplitude pitch around the leading edge
M	= undisturbed Mach number
M_1', M_2'	= real and imaginary parts of $C_m/(2k_r^2)$ relative to the leading edge, for a flat plate in unitary amplitude heave
M_3', M_4'	= real and imaginary parts of $C_m/(2k_r^2)$ relative to the leading edge, for unitary amplitude pitch around the leading edge
N	= number of panels
R	= hyperbolic radius
s	= Laplace transformation space
t	= time variable
U	= undisturbed flow velocity
W, w	= downwash on the airfoil surface, in real and transformed plane
X, Z	= physical plane position variables
$\bar{x}$	= equal to $(x - x_0)$
x_p	= center of rotation for pitch oscillation
x_{ref}	= reference point for moment calculation
x, z	= transformed plane position variables
x_0, z_0	= singularity position in transformed plane
α_0	= pitching motion amplitude
β	= compressibility factor, equal to $(M^2 - 1)^{1/2}$
Δx	= panel length
δ	= Dirac's function
μ	= doublet density
ν	= conormal direction
ξ	= perturbation velocity potential
σ	= source density
ϕ	= transformed velocity potential
ω	= angular frequency of oscillation
$*$	= convolution product; also, variable in s space

Introduction

In spite of the widespread use of panel methods to obtain solutions for flows ranging from incompressible to supersonic, practically no applications can be found for the two-dimensional supersonic case, either in steady or unsteady flow. The reason for this is the relative simplicity of this regime if one compares it, for example, with the subsonic case.

Satisfactory formulations for the problem of an airfoil in steady supersonic flow making small vertical harmonic oscillations have been available since the 1940s. The solution of Garrick and Rubinow[1] (also see Ref. 2) for that problem is simple and results in a numerical quadrature involving Bessel's function of the first kind and zeroth order and the known downwash distribution along the airfoil chord.

The arbitrary unsteady motion of a thin airfoil can also be represented by placing an appropriate sheet of sources along the airfoil chord. Alternatively, the problem can be analyzed by means of a Laplace transformation in the independent variable in the streamwise direction, as can be seen in Refs. 2 and 3.

The present investigation was stimulated by the statement by Garrick[4] that the disturbance field corresponding to compression or expansion over an airfoil in supersonic flow can be represented

Received April 15, 1994; presented as Paper 94-1891 at the AIAA Applied Aerodynamics Conference, Colorado Springs, CO, June 20–23, 1994; revision received Feb. 21, 1995; accepted for publication Feb. 24, 1995.

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by the action of doublets, although sources alone may suffice (which leads, in fact, to the usual way to attack the problem).

In the present work, the mathematical model for two-dimensional unsteady supersonic flow based on the reduced potential equation, also known as hyperbolic Helmholtz equation,[5] is presented and discussed. The purpose is to develop a rigorous formulation that shows the correspondence between supersonic and subsonic panel method theory.

Morino[6] obtained an integral solution to the steady supersonic flow problem through the application of Green's third theorem and showed its relationship with Ackeret's classical solution. Here, the same is done in order to produce an unsteady solution. It is shown that the source-only formulation is equivalent to that presented in Ref. 1, and that the doublet-only formulation reduces to a Volterra integrodifferential equation of the second kind. The relationship between source and doublet formulations is derived by Laplace transformation, which demonstrates that, in fact, the source-only formulation is a solution to the Volterra equation.

Finally, a numerical method is proposed in order to solve the Volterra integrodifferential equation. An extensive literature search could not produce a solution that uses a doublet panel method. The main purpose of this work is to propose such a solution. Computations are shown for the classical problems of a flat plate in harmonic heave and pitch and under a sinusoidal gust.

Mathematical Model

In a reference frame that translates steadily with the undisturbed flow velocity U, the perturbation velocity potential ξ due to the small-amplitude motion of a thin airfoil is governed by the linear convected wave equation,

$$\frac{\partial^2\xi}{\partial X^2}+\frac{\partial^2\xi}{\partial Z^2}-\frac{1}{a^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial X}\right)^2\xi=0 \tag{1}$$

U lies in the positive X direction, and ξ is nondimensional relative to (UL).

For a harmonically oscillating airfoil in steady flow, the perturbation velocity potential may be written as

$$\xi(X,Z,t)=\exp(i\omega t)\cdot\hat{\phi}(X,Z) \tag{2}$$

Introduction of Eq. (2) into Eq. (1) results in

$$\beta^2\frac{\partial^2\hat{\phi}}{\partial X^2}-\frac{\partial^2\hat{\phi}}{\partial Z^2}+\left(\frac{2i\omega U}{a^2}\right)\frac{\partial\hat{\phi}}{\partial X}-\frac{\omega^2}{a^2}\hat{\phi}=0 \tag{3}$$

where $\beta^2=M^2-1$ and M is the undisturbed flow Mach number.

Defining a new complex potential

$$\phi=\exp[i\omega UX/(a\beta)^2]\cdot\hat{\phi} \tag{4}$$

and using the Prandtl–Glauert transformation,

$$x=X/L;\qquad z=\beta Z/L \tag{5}$$

Eq. (3) can be rewritten as the hyperbolic Helmholtz equation, i.e.,

$$\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial z^2}+K^2\phi=0 \tag{6}$$

where

$$K=\omega L/a\beta^2 \tag{7}$$

When K is compared with the reduced frequency,

$$k_r=\omega L/U \tag{8}$$

one can readily show that

$$K=k_rM/(M^2-1) \tag{9}$$

Notice that the term in K^2 determines how much the hyperbolic Helmholtz equation (6) departs from the wave equation, which governs steady flow. In fact, quasisteady solutions can be found by using the wave equation and introducing unsteadiness through the boundary conditions. This is, indeed, a good approximation for high Mach numbers and low reduced frequencies. However, as illustrated in Fig. 1, for Mach numbers slightly greater than 1.0 or for high frequencies, it appears to be important to include the unsteady term in the equation because of the magnitude of K.

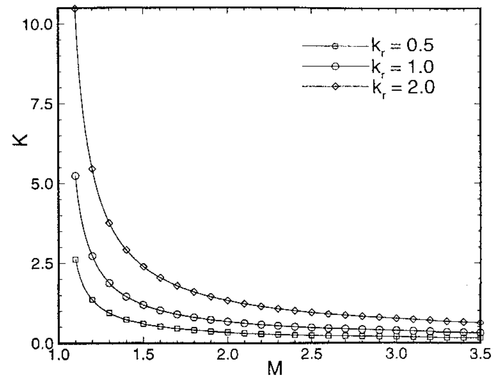

Fig. 1 Variation of factor K in the hyperbolic Helmholtz equation with Mach number M and the reduced frequency k_r.

The problem description is completed with the small-disturbance boundary condition of zero relative normal velocity on the airfoil surface. Consider the case of a flat plate. In the transformed plane, the surface boundary condition translates into

$$\frac{\partial\phi}{\partial z}=\frac{\exp(iKMx)}{\beta}\left(\frac{\partial h}{\partial x}+ik_rh\right) \tag{10}$$

where $h(x)$ is the nondimensional line that represents the plate surface.

After solution of Eq. (6) for ϕ, subjected to boundary condition (10), the pressure coefficient on the airfoil surface is obtained from

$$C_p(x)=-2\exp(-iKMx)\left(\frac{\partial\phi}{\partial x}-\frac{iK}{M}\phi\right) \tag{11}$$

and $C_p(x,t)=C_p(x)\exp(i\omega t)$.

A similar mathematical model for subsonic flow leading to the Helmholtz equation is presented in Ref. 7.

Boundary Integral Formulation

Consider the hyperbolic Helmholtz equation (6). The fundamental unitary-strength source and doublet (in the z direction) solutions for this equation, at point (x,z), are

$$\phi_S(x,z)=-\tfrac{1}{2}J_0(KR) \tag{12}$$

$$\phi_D(x,z)=\frac{1}{2}\left[\frac{K(z-z_0)}{R}J_1(KR)-\delta(R)\right] \tag{13}$$

respectively, where

$$R=\sqrt{(x-x_0)^2-(z-z_0)^2} \tag{14}$$

The second term on the right-hand side of the doublet solution (Dirac's delta function) comes from the z derivative of the source solution at $R=0$, which represents the Mach lines that originate at (x_0,z_0) due to the presence of the point source, and run to the upper and lower right.

The application of Green's third theorem to the hyperbolic Helmholtz equation is described in the Appendix. For an airfoil on the x axis (i.e., $z_0=0$), with leading-edge at $x=-1$, it yields the following integral identity for the upper flow region ($z>0$),

$$\phi(x,z)=-\int_{-1}^{x-z}\left(\frac{\partial\phi}{\partial z}\right)_{\text{upper}}J_0(KR)\,\mathrm{d}x_0+\int_{-1}^{x-z}(\phi)_{\text{upper}}\times\frac{Kz}{R}J_1(KR)\mathrm{d}x_0-\int_{-1}^{x-z}(\phi)_{\text{upper}}\delta(\bar{x}-z)\,\mathrm{d}x_0 \tag{15}$$

where $\tilde{x} = x - x_0$, and similarly for the lower region. Notice the appearance of source and doublet integrals, as in the subsonic case.

For a source-only formulation, only the first integral survives, providing a simple solution to Eq. (6) on the airfoil surface, namely

$$\phi(x) = -\int_{-1}^{x} \sigma(x_0)\frac{1}{2} J_0[K(x - x_0)]\,dx_0 \tag{16}$$

This is equivalent to the solution for the thin airfoil in small vertical harmonic motion presented in Ref. 1. One recognizes $\sigma(x_0)$ as the surface density of the first fundamental solution, see Eq. (12). The solution is completed with the application of the boundary condition (10). For a thin airfoil, the density of the sources distributed over the chord is given by

$$\sigma(x) = 2w(x) = 2\frac{\partial\phi}{\partial z}(x) \tag{17}$$

This result is analogous to the one obtained for the thin-airfoil thickness problem in subsonic or supersonic steady flow. Observe, however, that the airfoil surface slope $w(x)$ and, consequently, the source density itself may be different for the upper- and lower-surface solutions.

For a doublet-only formulation, only the last two integrals in Eq. (15) remain. Differentiation in z and introduction of the boundary condition lead to

$$\frac{\partial\phi}{\partial z}(x) = -\int_{-1}^{x} \mu(x_0)\frac{K}{2(x - x_0)} J_1[K(x - x_0)]\,dx_0 - \frac{1}{2}\frac{\partial\mu}{\partial x}(x) \tag{18}$$

which is a Volterra integrodifferential equation of the second kind. Again, one recognizes $\mu(x)$ as the surface density of the doublet fundamental solution, defined in Eq. (13), obtained from

$$\mu(x) = 2\phi(x) \tag{19}$$

where μ is placed on the chord and ϕ is taken on the airfoil surface.

The integral equation presented in Eq. (18) has the same form as in the subsonic airfoil camber problem, except for the second right-hand-side term. In fact, the classical direct and inverse problems of airfoil theory are fully retrieved from Eqs. (16) and (18).

A remarkable result can be obtained when Eq. (18) is solved by Laplace transformation. After introducing relation (19), that integral equation can be rewritten as

$$w(x) = -K\phi(x) * \frac{1}{x} J_1(Kx) - \frac{\partial\phi}{\partial x}(x) \tag{20}$$

where the asterisk indicates the convolution product.

Moving the airfoil leading edge to $x = 0$ and taking the Laplace transform with respect to x, one turns Eq. (20) into

$$w^*(s) = \phi^*(s)\left[(s^2 + K^2)^{\frac{1}{2}} - s\right] - s\,\phi^*(s) \tag{21}$$

where variables with asterisks have the usual sense in s space. Rearranging, one finds

$$\phi^*(s) = -w^*(s)\Big/\left[(s^2 + K^2)^{\frac{1}{2}}\right] \tag{22}$$

This relation, after inverse Laplace transformation, becomes

$$\phi(x) = -\int_{-1}^{x} w(x_0) J_0[K(x - x_0)]\,dx_0 \tag{23}$$

returning the same result as in Eq. (16). This shows that the source integral is a solution to the doublet equation (18). In other words, Eq. (18) is the true boundary integral equation equivalent to the system formed by the hyperbolic Helmholtz differential equation (6) and the boundary condition (10).

Panel Method Solution

In order to obtain a panel method solution for the doublet-only formulation [Eqs. (18) and (19)], the x axis between -1 and 1 (i.e., the airfoil chord) is uniformly divided into a number N of small elements. Contrary to the subsonic case, where control points are normally chosen at the center of each panel, the hyperbolic nature of the integrodifferential equation for supersonic flow suggests upwind bias.

For $x = -1$, Eq. (18) reduces to

$$\frac{\partial\phi}{\partial z}(-1) = -\frac{1}{2}\frac{\partial\mu}{\partial x}(-1) = \frac{\partial\phi}{\partial x}(-1) \tag{24}$$

where the left-hand side is known from the boundary condition, Eq. (10). Additionally, for $x < -1$,

$$\phi = 0, \qquad \frac{\partial\phi}{\partial x} = 0 \tag{25}$$

The integration of the solution can start at point $x = -1$, since the values for the potential and its first derivative are known there. That suggests the choice of such point, the leading edge of the first panel, as the first control point. Therefore, the control points are placed at the leading edge of each panel, a scheme that is compatible with an upwind bias in the solution. This was the only scheme for placement of control points that was studied, and a different scheme would probably require a different numerical method of solution.

The numerical solution is composed of the values for doublet density and, consequently, potential at the control points. Velocities and pressures can be computed using central finite differences.

The discretized form of Eq. (18) is, on an arbitrary control point $i > 1$,

$$\left(\frac{d\phi}{dx}\right)_i = -w(x_i) - \sum_{j=2}^{i}\frac{1}{2}[I_{i,j} + I_{i,j-1}] \tag{26}$$

$$I_{i,j} = \phi_j \frac{K\,J_1[K(x_i - x_j)]}{x_i - x_j}\Delta x$$

The perturbation velocity potential for a control point $(i + 1) > 3$ is then computed using Adam's third-order formula, namely,

$$\phi_{i+1} = \phi_i + \frac{\Delta x}{12}\left[23\left(\frac{d\phi}{dx}\right)_i - 16\left(\frac{d\phi}{dx}\right)_{i-1} - 5\left(\frac{d\phi}{dx}\right)_{i-2}\right] \tag{27}$$

The value of ϕ_1 (at $x = -1$) is zero. The values for ϕ_2 and ϕ_3 are obtained using Adam's first- and second-order formulas, respectively.

The procedure described is applied for each control point until the airfoil trailing edge ($x = 1$) is reached. Observe that the preceding method is, as in the subsonic case, equivalent to the assembly of a linear matrix equation, given by

$$[A]\{\mu\} = \{B\} \tag{28}$$

where $[A]$ is the influence coefficient matrix, $\{\mu\}$ is the vector containing the doublet density values at the panel control points, and $\{B\}$ contains the normal velocity component at each control point w. The main difference from the subsonic case is in the fact that matrix $[A]$ is lower triangular, as a consequence of the upwind dependency of supersonic phenomena. The computational method has operation count of order N^2.

Results

The doublet-only method was used to compute three basic unsteady flows applied to a flat plate: harmonic heave and pitch, and sinusoidal gust. Because of the linearity of the formulation, these three solutions can be combined to produce more complex airfoil motions.

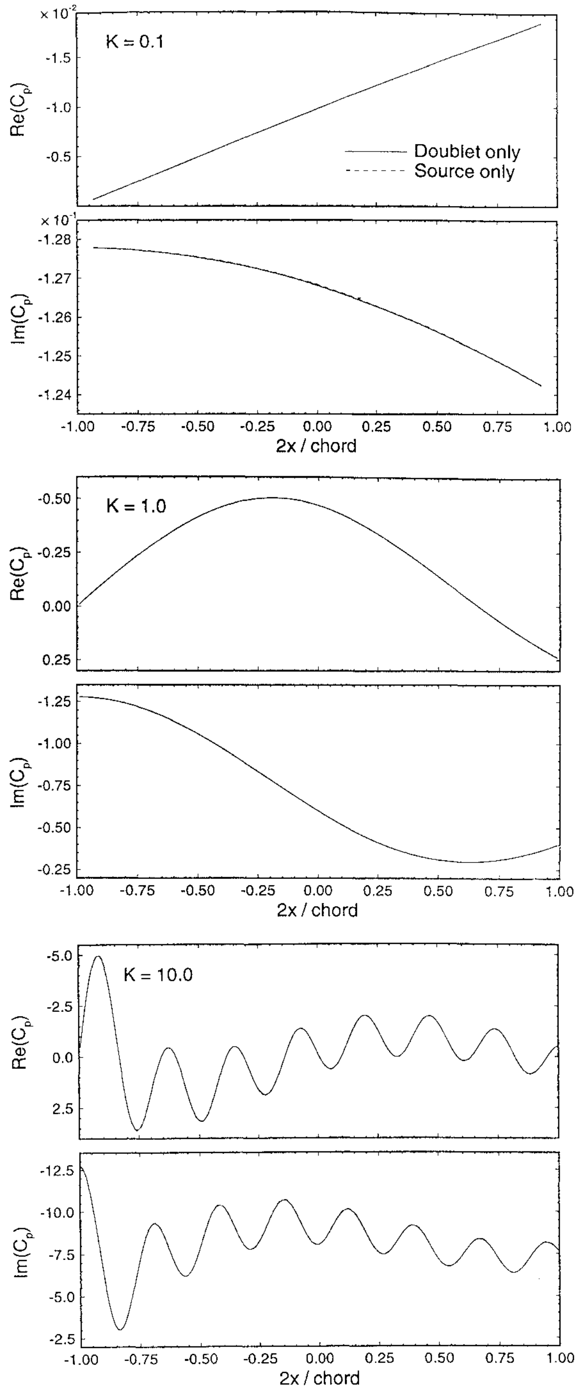

Fig. 2 Real and imaginary parts of the pressure coefficient for a flat plate in heaving motion, for different values of K; Mach number is 1.3.

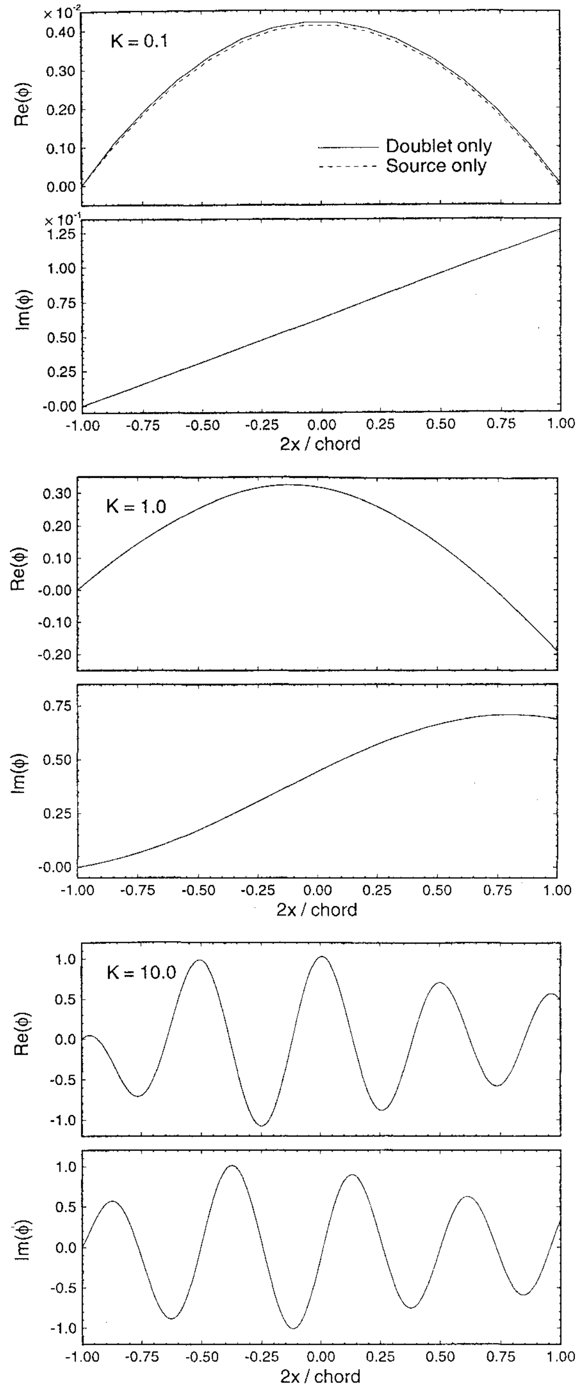

Fig. 3 Real and imaginary parts of the potential for a flat plate in heaving motion, for different values of K; Mach number is 1.3.

The purposes of the computations were 1) to validate the present method through comparisons with previous solutions, 2) to evaluate the accuracy of the panel method solution and its relation to discretization, and 3) to evaluate the accuracy of quasisteady formulations when compared to the unsteady solutions.

Harmonic Heave and Pitch

For a flat plate in harmonic heave, the vertical positions of all its points vary according to

$$h(x, t) = h(x)\exp(i\omega t) = -h_0 \exp(i\omega t) \tag{29}$$

For harmonic pitch around point x_p, the angle of attack varies according to

$$\alpha = \alpha_0 \exp(i\omega t) \tag{30}$$

which means that the plate surface location is given by

$$h(x, t) = -\alpha_0(x - x_p)\exp(i\omega t) \tag{31}$$

Figures 2 and 3 show C_p and ϕ distributions along the chord for unitary heave ($h_0 = 1$). A low Mach number of 1.3 was fixed, and three frequencies were chosen, corresponding to $K = 0.1$, 1.0, and 10.0. As the frequency increased, the discretization had to be refined to capture the oscillatory behavior of the solution. The number of panels used for each solution were 15, 100, and 200, respectively, chosen somewhat arbitrarily. Results obtained with both the doublet-only and source-only formulations are shown. Observe that the differences between the two panel solutions are discernible only in the lower frequency case, in which a very crude discretization was used. Such differences disappear quickly as the discretization is increased, for constant frequency.

Table 1 Values of lift and moment for a heaving flat plate, computed by doublet-only panel method

M	k_r	L_1	L_2	M'_1	M'_2
10/9	0.0152	8.75727	135.136	11.6644	134.786
	0.1805	4.24468	6.22881	4.62868	4.08162
	1.9000	−0.252511E-01	0.445591	−0.755672E-01	0.463411
5/3	0.0128	0.421752	58.5853	0.562304	58.5811
	0.1600	0.403053	4.58479	0.532424	4.53389
	1.6000	−0.197075E-01	0.360965	−0.560248E-01	0.373388
2.5	0.0168	0.831076E-01	25.9767	0.110804	25.9758
	0.1596	0.810678E-01	2.71898	0.107542	2.71122
	2.1000	−0.100349E-01	0.190802	−0.200412E-01	0.197768
5.0	0.0192	0.850288E-02	10.6313	0.113366E-01	10.6312
	0.1632	0.834067E-02	1.24933	0.110771E-01	1.24862
	2.4000	−0.147088E-02	0.836853E-01	−0.270296E-02	0.847734E-01

Table 2 Values of lift and moment for a flat plate in pitching motion

M	k_r	L'_3	L'_4	M'_3	M'_4
10/9	0.0152	8896.41	−440.649	8876.27	−586.836
	0.1805	38.3694	−15.1402	28.1102	−15.1781
	1.9000	0.259593	0.441060	0.249426	0.609379
5/3	0.0128	4577.26	25.6401	4577.07	34.1883
	0.1600	28.9286	2.11660	28.7461	2.83956
	1.6000	0.242213	0.360859	0.239358	0.492409
2.5	0.0168	1546.29	21.0306	1546.26	28.0410
	0.1596	17.0908	2.21879	17.0693	2.95974
	2.1000	0.908296E-01	0.188615	0.898127E-01	0.253671
5.0	0.0192	553.719	10.1885	553.717	13.5847
	0.1632	7.66081	1.19894	7.65923	1.59866
	2.4000	0.346301E-01	0.832100E-01	0.344790E-01	0.111194

The lift and moment coefficients for the airfoil are given by

$$C_l(x,t) = [C_{l1}(x) + iC_{l2}(x)]\exp(i\omega t)$$

$$= \int_{-1}^{1}\left(C_{p_{\text{lower}}} - C_{p_{\text{upper}}}\right)\frac{1}{2}\mathrm{d}x \tag{32}$$

$$C_m(x,t) = [C_{m1}(x) + iC_{m2}(x)]\exp(i\omega t)$$

$$= \int_{-1}^{1}\left(C_{p_{\text{lower}}} - C_{p_{\text{upper}}}\right)\frac{x - x_{\text{ref}}}{4}\mathrm{d}x \tag{33}$$

Another description for the real and imaginary parts of these coefficients is usual[1] and defined as follows:

$$C_l(x) = 4k_r^2[L_1(x) + iL_2(x)] \tag{34}$$

$$C_m(x) = 2k_r^2[M'_1(x) + iM'_2(x)] \tag{35}$$

for heave, and

$$C_l(x) = 4k_r^2[L'_3(x) + iL'_4(x)] \tag{36}$$

$$C_m(x) = 2k_r^2[M'_3(x) + iM'_4(x)] \tag{37}$$

for pitch around the leading edge. The reference point for moments is also the leading edge. Nose-down pitching moments are defined as positive, in agreement with Ref. 1.

Tables 1 and 2 show values for lift and moment computed by the doublet-only method, for a flat plate in unitary heave ($h_0 = 1$) and pitch ($\alpha_0 = 1$), respectively. Several Mach numbers and frequencies were chosen for each case. The tabulated values were converged, which means that the number of panels was increased until the solution did not change for the six significant digits printed. When these values were compared with those computed by Garrick and Rubinow,[1] most were identical and the maximum deviation was 0.01%. Some of the computations required numbers of panels of the order of 2000, but even those cases did not exceed 10-s CPU time in a IBM-RS6000/340 workstation.

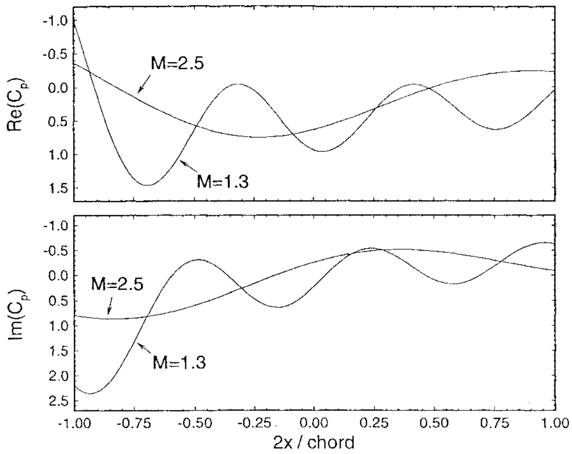

Fig. 4 Pressure coefficient distribution for a flat plate under sinusoidal gust, for k_r = 2.0.

Sinusoidal Gust

For a flat plate under a sinusoidal gust, the boundary condition at the surface should not be expressed in terms of the surface curve $h(x)$. Instead, one considers the local downwash at the chord and returns to the linearized condition of zero normal velocity at the plate surface,

$$\frac{\partial \xi}{\partial Z}(X,t) = \frac{W(X,t)}{UL} \tag{38}$$

in real space. The velocity perturbation produced by the gust is given by

$$W(X,t) = w_g U\exp(-i\omega X/U)\exp(i\omega t) \tag{39}$$

In transformed space, the boundary condition takes the form

$$\frac{\partial \phi}{\partial z} = \frac{\exp(iKMx)}{\beta}w_g\exp(-ik_r x) \tag{40}$$

Figure 4 show values for the C_p distribution along the chord for a flat plate under a unit-strength ($w_g = 1$) sinusoidal gust, computed with the doublet-only method. A frequency corresponding to

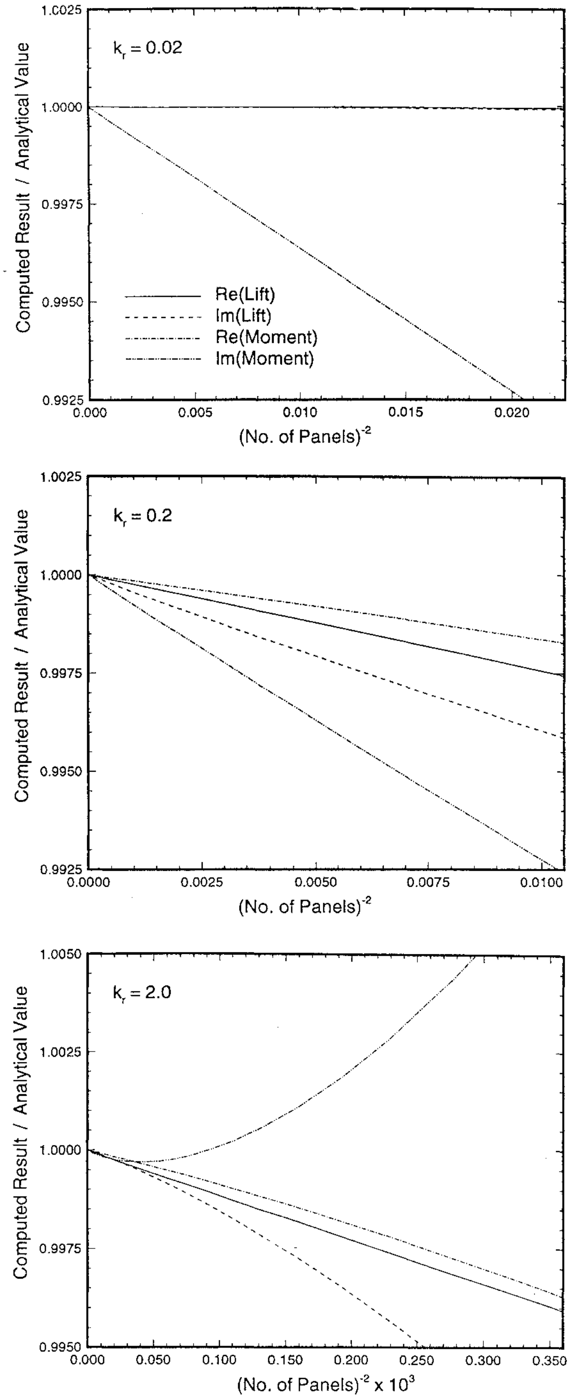

Fig. 5 Variation of lift and moment with discretization for a flat plate under sinusoidal gust, doublet-only solution, for $M = 1.3$.

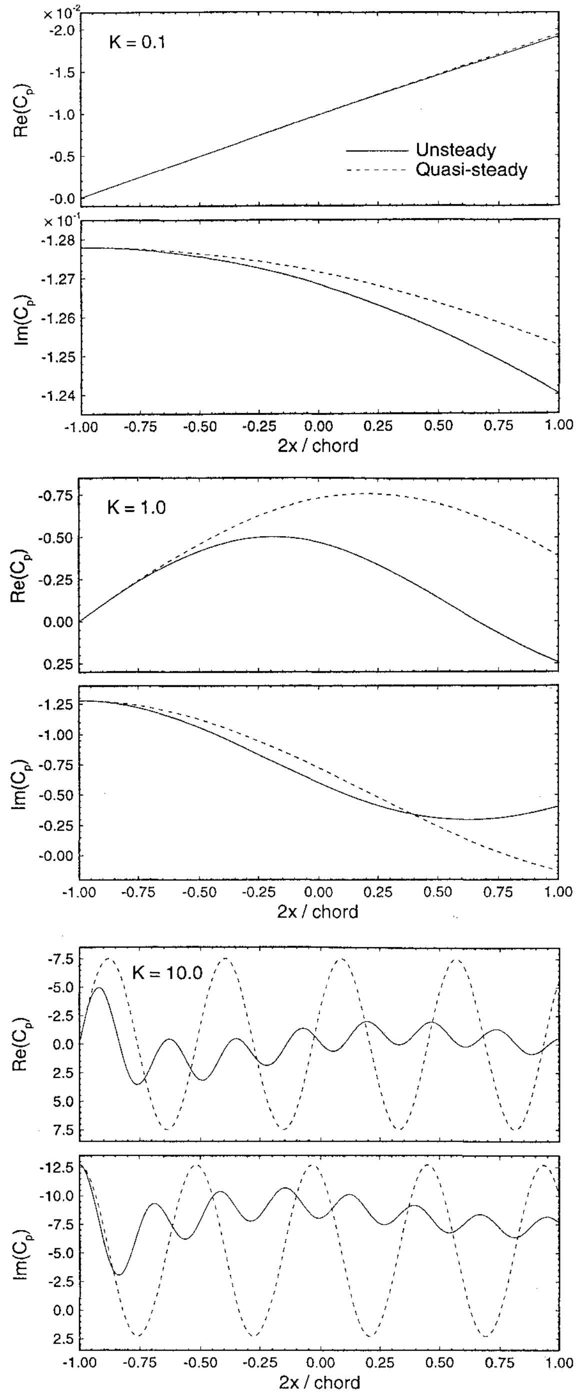

Fig. 6 Comparison of unsteady and quasisteady solutions for pressure coefficient distribution on a heaving flat plate, for $M = 1.3$.

$k_r = 2.0$ was chosen, and the Mach numbers were fixed at 1.3 and 2.5. The number of panels used was 500, which guaranteed that load values were converged until a 0.01% precision or less. The reader should note that since the solutions were very cheap computationally the discretization was generally taken to higher values than necessary for accuracy. Notice that for the same frequency the solution becomes less oscillatory as the Mach number increases. The heave and pitch problems have the same behavior.

Solutions for the gust problem were used to assess the accuracy and convergence properties of the doublet method. The computed values were compared with the analytical solution (see Ref. 2), given by

$$C_{l_g} = -(4w_g/\beta)\,\exp(ik_r)\,f_0(M, K) \tag{41}$$

$$C_{m_g} = 2w_g/\beta\,\exp(ik_r)[(x_{\text{ref}} + 1)f_0(M, K) - 2f_1(M, K)] \tag{42}$$

(The equation for C_m presented in Ref. 2 had to be corrected, due to an error in the terms in f_0 and f_1.) The f_λ integrals are defined by

$$f_\lambda(M, K) = \int_0^1 x^\lambda \exp(-2iKMx)J_0(2Kx)\,\mathrm{d}x \tag{43}$$

The problem was solved with the doublet method for $M = 1.3$ and three frequencies, using several discretizations for each case. The results are shown in Fig. 5. The ratios between computed and analytical results for each of the parts (real, imaginary) of the lift and moment coefficients are plotted against N^{-2}. Notice that the x-axis scaling varies for each plot; cases with higher frequencies require a larger number of panels to maintain the same accuracy. A 0.01% accuracy could be achieved with around 60, 90, and 400 panels for frequencies corresponding to $k_r = 0.02$, 0.2, and 2.0, respectively. It is interesting to observe that, for the lower frequency case, three

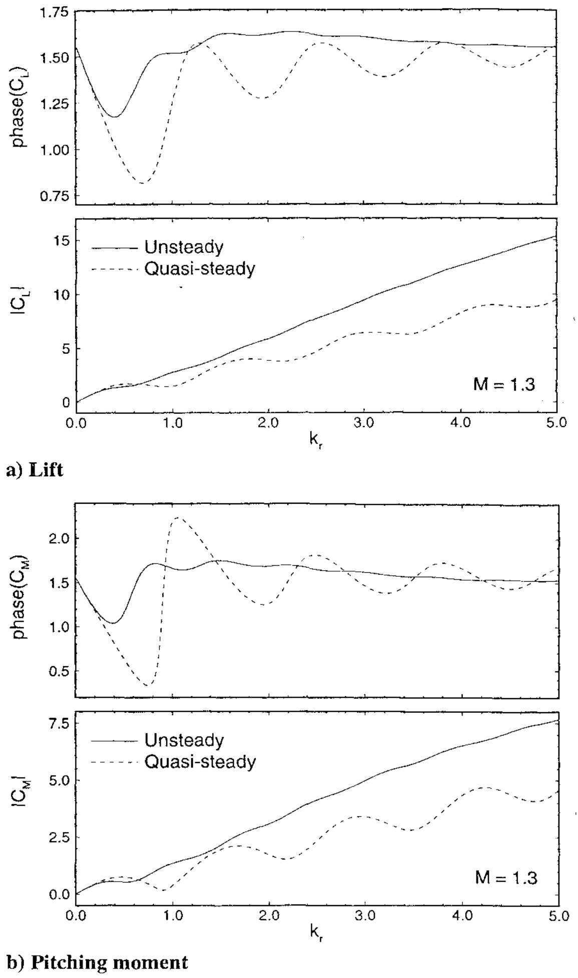

Fig. 7 Comparison of unsteady and quasisteady solutions for a heaving flat plate at M = 1.3.

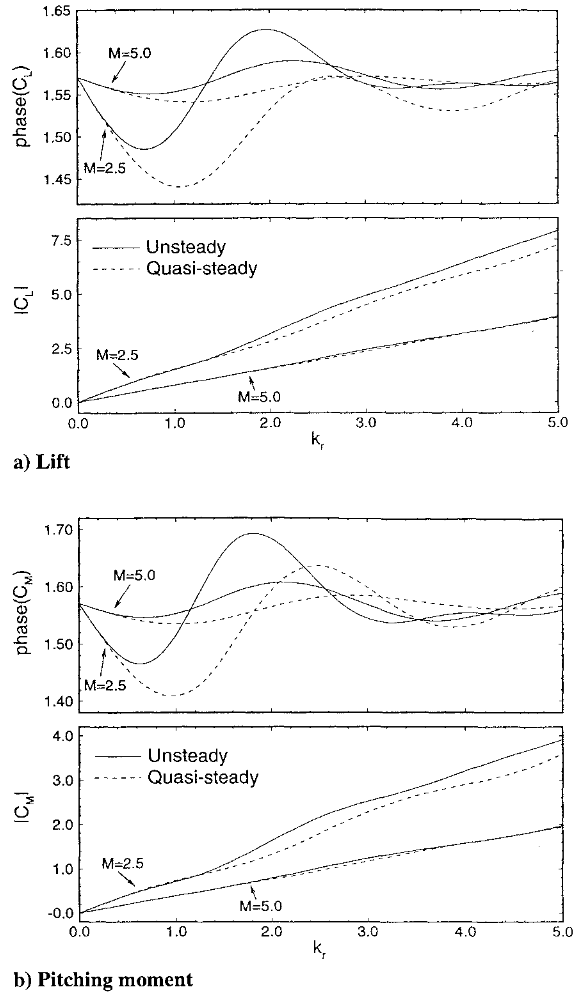

Fig. 8 Comparison of unsteady and quasisteady solutions for a heaving flat plate.

of the computed terms were essentially exact even for very coarse discretizations.

Quasisteady Solution Comparison

A simple way to capture first-order effects of unsteadiness in a potential flow calculation is to retain only the unsteady boundary conditions, while solving the steady equation. This is useful, for instance, if one has access to a powerful steady solver but needs an estimate of parameters for an unsteady case. This technique has been used widely for supersonic flow computations, especially since, in general, the frequencies of interest can be considered low. The purpose of the following computations was to assess the behavior, both quantitative and qualitative, of the quasisteady solution when compared with the full unsteady result.

In the quasisteady approach to the present problem, the frequency-related factor K is set to zero in Eq. (6), reducing it to the wave equation. The boundary condition (10) remains unchanged.

Figure 6 compares the unsteady and quasisteady solutions for C_p along the chord for a (unit-strength) heaving flat plate. The Mach number was fixed at 1.3, and the frequencies chosen corresponded to K = 0.1, 1.0, and 10.0. The doublet-only method was used with 200 panels. Such a number essentially eliminated the effect of discretization for the purpose of C_p plot comparison. As expected, for low frequencies the quasisteady computation is a good approximation to the unsteady solution. As the frequency increases, the approximation departs markedly from the correct result; notice that for the case K = 10.0 the quasisteady solution (a pure harmonic) is very different qualitatively from the unsteady one.

The next step was to examine the difference in integrated coefficient values for the two computations. A frequency sweep in k_r was performed for M = 1.3, 2.5, and 5.0 using 500 panels. Once again, this discretization guaranteed convergence until 0.01% or less. Although the curves show results until k_r = 5.0, frequencies corresponding to $k_r > 1$ can be considered unusual for aeroelastic problems. Figures 7 and 8 show the absolute value and phase angle of C_l and C_m as a function of frequency.

Considering Fig. 7, it is clear that, for a Mach number close to 1.0, the quasisteady approximation is not satisfactory for frequencies greater than $k_r \approx 0.3$. As the Mach number is increased, the results from the quasisteady approach improve. For M = 2.5 (Fig. 8), the quasisteady approximation has a maximum error of about 10% in phase and 20% in absolute value. For M = 5.0 the quasisteady results are even better: although the phase values are qualitatively wrong relative to the unsteady solution, the differences are of the order of 2%—observe that the phase angles of the coefficients are centered around 90 deg. The differences in absolute value are even smaller.

These results illustrate why the quasisteady approach has been so valuable in supersonic studies. Except for Mach numbers close to the transonic range (and higher reduced frequencies), the quasisteady results can be considered acceptable for engineering studies.

Conclusion

This work presents a rigorous formulation for the application of panel methods to unsteady supersonic flow. It shows that in analogy with subsonic flow there are source and doublet solutions to the problem. The apparent lack of interest so far for the doublet approach is due to the simplicity of the problem that allows the use of only sources, which produces a solution by a straightforward quadrature. In other words, whereas in the subsonic case source and doublets are necessary to solve the thickness and camber problems, respectively, in supersonic flow either one of those singularities can be used. It has been shown that the source and doublet integrals are both solutions to the hyperbolic Helmholtz differential equation applied to airfoil flow. The numerical implementation of both formulations demonstrated that the doublet method is as accurate and efficient as the source method.

Appendix: Green's Theorem Application

This appendix presents a brief description of the application of Green's third theorem to Eq. (6). It follows the same approach as in Ref. 8.

Let $\mathcal{S}$ be an arbitrary surface, bounded by a contour $\mathcal{C}$. If σ and ϕ are two functions which, together with their first and second derivatives, are finite and single valued throughout $\mathcal{S}$, then

$$\int\int_{\mathcal{S}} (\sigma\mathcal{L}\phi - \phi\mathcal{L}\sigma)\,\mathrm{d}S = \oint_{\mathcal{C}} \left(\sigma\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial\sigma}{\partial\nu}\right)\mathrm{d}\ell \tag{A1}$$

where $\mathcal{L}$ is the hyperbolic Helmholtz operator,

$$\mathcal{L} \equiv \frac{\partial}{\partial x^2} - \frac{\partial}{\partial z^2} + K^2 \tag{A2}$$

and ν is the conormal direction to contour $\mathcal{C}$, obtained by reversing the x component of the contour's normal direction. Making $\sigma \equiv J_0(KR)$ and taking ϕ as a solution to the hyperbolic Helmholtz equation, it follows that

$$\oint_{\mathcal{C}} \left[J_0(KR)\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial J_0(KR)}{\partial\nu}\right]\mathrm{d}\ell = 0 \tag{A3}$$

Consider now Fig. A1. A contour $\mathcal{C}_1$ is chosen so that it is composed of five joined segments, namely, OA, AP, PB, BD and DO. Segments OB and AP are parts of upper Mach lines originating at O and A, respectively. On the other hand, segments BP and DA are parts of lower Mach lines passing through P and A. Another contour $\mathcal{C}_2$ is formed by OA, AD, and DO. Equation (A3) is then applied to each one of the contours. When the resulting expressions are simplified and subtracted, one is left with

$$\phi(x,z) = -\int_0^{x-z} J_0(KR)\frac{\partial\phi}{\partial z}\mathrm{d}x_0 + \int_0^{x-z} \phi\frac{\partial J_0(KR)}{\partial z}\,\mathrm{d}x_0 \tag{A4}$$

where $\phi(x, y)$ is the value of ϕ on P. Notice that the integrals are along the x axis, so that $z_0 = 0$.

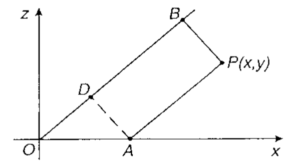

Fig. A1 Regions for Green's theorem integration.

Careful evaluation of the last integral on the right-hand side of Eq. (A4) leads to the final result,

$$\phi(x,z) = -\int_0^{x-z} \frac{\partial\phi}{\partial z} J_0(KR)\,\mathrm{d}x_0 + \int_0^{x-z} \phi\left[\frac{Kz}{R}J_1(KR) - \delta(\bar{x} - z)\right]\mathrm{d}x_0 \tag{A5}$$

where the Dirac's function term appears when $J_0(KR)$ is derived, in the sense of distribution theory, along z for $R = 0$.

Observe that the ϕ and $\partial\phi/\partial z$ terms in the integrals of Eq. (A5) are evaluated on the upper side of the x axis. For P in the $z < 0$ plane, a similar result is found, but the ϕ terms come from the lower side of the axis, and the signs of the integrals are reversed.

Acknowledgment

This work was partially supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brasília, DF), under Grant 300.682/93-0.

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